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Study of the change from walking to non-walking behavior in a vectorial gauge theory as a function of N_f

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Abstract: We study a vectorial gauge theory with gauge group $\mathrm{SU}(N_c)$ and a variable number, N_f, of massless fermions in the fundamental representation of this group. Using approximate solutions of Schwinger-Dyson and Bethe-Salpeter equations, we calculate meson masses and investigate how these depend on N_f. We focus on the range of N_f extending from near the boundary with a non-Abelian Coulomb phase, where the theory exhibits a slowly running ("walking") gauge coupling, toward smaller values where the theory has non-walking behavior. Our results include determinations of the masses of the lowest-lying flavor-adjoint mesons with $J^{\mathrm{PC}} = 0^{-+}$, 1^{--}, 0^{++}, and 1^{++} (the generalized π, ρ, a_0, and a_1). Related results are given for flavor-singlet mesons and for the generalization of f_π. These results give insight into the change from walking to non-walking behavior in a gauge theory, as a function of N_f.

Keywords: Spontaneous Symmetry Breaking; Confinement; Technicolor and Composite Models.

Contents

1. Introduction

We consider a (3+1)-dimensional vectorial gauge theory (at zero temperature and chemical potential) with the gauge group $\mathrm{SU}(N_c)$ and N_f massless fermions transforming according to the fundamental representation of this group. For $N_c = 3$, if one took $N_f = 2$, this would be an approximation to actual quantum chromodynamics (QCD) with just the u and d quarks, since their current quark masses are small compared with the scale $\Lambda_{\mathrm{QCD}} \simeq 400\,\mathrm{MeV}$. We restrict here to the range $N_f < (11/2)N_c$ for which the theory is asymptotically free. An analysis using the two-loop beta function and Schwinger-Dyson equation leads to the inference that for N_f in this range, the theory includes two phases: (i) for $0 \le N_f \le N_{f,cr}$ a phase with confinement and spontaneous chiral symmetry breaking (SχSB); and (ii) for $N_{f,cr} \le N_f \le (11/2)N_c$ a non-Abelian Coulomb phase with no confinement or spontaneous chiral symmetry breaking. We shall refer to $N_{f,cr}$, the critical value of N_f, as the boundary of the non-Abelian Coulomb (conformal) phase [1]. Here we take electroweak interactions to be turned off. We denote the fermions as f_i^a with $a = 1, \ldots, N_c$ and $i = 1, \ldots, N_f$. The theory has an $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \times \mathrm{U}(1)_V$ global symmetry (the $\mathrm{U}(1)_A$ being explicitly broken by instantons), which is spontaneously broken to $\mathrm{SU}(N_f)_V \times \mathrm{U}(1)_V$ by the formation of a bilinear fermion condensate.

For N_f slightly less than $N_{f,cr}$, the theory exhibits an approximate infrared (IR) fixed point with resultant walking behavior. That is, as the energy scale μ decreases from large values, $\alpha = g^2/(4\pi)$ (g being the $\mathrm{SU}(N_c)$ gauge coupling) grows to be O(1) at a scale Λ, but increases only rather slowly as μ decreases below this scale, so that there is an extended interval in energy below Λ where α is large, but slowly varying. Associated with this slowly running behavior, the resultant dynamically generated fermion mass, Σ, is much smaller than Λ. In addition to its intrinsic field-theoretic interest, this walking behavior has played

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an important role in theories of dynamical electroweak symmetry breaking [2]–[7]. As N_f approaches $N_{f,cr}$ from below, quantities with dimensions of mass vanish continuously; i.e., the chiral phase transition separating phases (i) and (ii) is continous. Recently, meson masses and other quantities such as the generalized pseudoscalar decay constant f_π were calculated in the walking limit of an SU(N_c) gauge theory [8].

In the present paper we shall investigate how meson masses and other quantities change as one decreases N_f below $N_{f,cr}$, moving away from the boundary, as a function of N_f, between phases (i) and (ii), deeper into the confined phase. Our paper is thus a study of the change (crossover) between the walking behavior that occurs near to this boundary, and the non-walking behavior that occurs for smaller N_f. In a non-walking (asymptotically free, confining) theory such as real QCD, as the energy scale μ decreases through Λ, α increases rapidly through values of order unity, triggering spontaneous chiral symmetry breaking on this scale, so that $\Sigma \sim \Lambda$. This is quite different from a theory with walking, in which $\Sigma \ll \Lambda$. As in ref. [8], we use the Schwinger-Dyson (SD) equation to compute the dynamical fermion mass Σ (generalized constituent quark mass) and then insert this into the Bethe-Salpeter (BS) equation to obtain the masses of the low-lying mesons. We restrict to an interval of N_f values for which the theory has an infrared fixed point (as calculated from the beta function, to be discussed further below). The reason for this is that it makes our calculations more robust since for our interval of N_f we can avoid having to introduce a cutoff on the growth of α in the infrared. If one uses Schwinger-Dyson and Bethe-Salpeter equations to explore a region of N_f where the beta function does not have an infrared fixed point, one must use such an IR cutoff, which leads to cutoff-dependence of the results. Some related work is in [9]–[12]. For definiteness, we shall take $N_c = 3$; however, as will be seen, N_c only enters indirectly, via the dependence of the value of the infrared fixed point α_* on N_c. Hence, our findings may also be applied in a straightforward way, with appropriate changes in the value of α_*, to an SU(N_c) gauge theory with a different value of N_c.

This paper is organized as follows. In section II we review some background material concerning the beta function, approximate infrared fixed point, and walking behavior. Section III includes a discussion of the Schwinger-Dyson equation and our solution of it, as well as our calculation of the pseudoscalar decay constant f_P, the generalization of f_π. In section IV we present our calculation of meson masses using the Bethe-Salpeter equation. Section V contains our conclusions.

2. Preliminaries

In order to study meson masses and other quantities as one moves away from the boundary between the confined phase with spontaneous chiral symmetry breaking and the non-Abelian Coulomb phase, it is first necessary to know as accurately as possible where this boundary lies, as a function of N_f, i.e., to know the value of $N_{f,cr}$. For sufficiently large N_f, the beta function (calculated to the maximal scheme-independent order, namely two-loops) has an IR fixed point at

$$\alpha_* = \frac{-4\pi(11N_c - 2N_f)}{34N_c^2 - 13N_cN_f + 3N_c^{-1}N_f} \ . \tag{2.1}$$

Requiring that α_* be sufficiently large as to yield spontaneous symmetry breaking in the context of an approximate solution to the Schwinger-Dyson gap equation for the inverse propagator of a fermion transforming according to the fundamental representation of $SU(N_c)$ yields the condition that $N_f < N_{f,cr}$, where [7]

$$N_{f,cr} = \frac{2N_c(50N_c^2 - 33)}{5(5N_c^2 - 3)} \ . \tag{2.2}$$

For $N_c = 3$ this gives $N_{f,cr} \simeq 11.9$. These estimates are only rough, in view of the strongly coupled nature of the physics. Effects of higher-order gluon exchanges and instantons have been studied in refs. [13]. In principle, lattice gauge simulations provide a way to determine $N_{f,cr}$, but the groups that have studied this have not reached a consensus [14].

In our analysis, what we actually vary is the value of the approximate IR fixed point α_*, which depends parametrically on N_f. Thus, although our SD and BS equations are semi-perturbative, the analysis is self-consistent in the sense that our α_{cr} really is the value at which, in our approximation, one passes from the confinement phase to the non-Abelian Coulomb phase, and our values of α do span the interval over which there is a crossover from walking to QCD-like (i.e., non-walking) behavior.

As is evident from the above results, decreasing N_f below $N_{f,cr}$ has the effect of increasing α_* and thus moving the theory deeper in the phase with confinement and spontaneous chiral symmetry breaking, away from the boundary with the non-Abelian Coulomb phase. This is the key parametric dependence that we shall use for our study. Our aim is to investigate how meson masses and other observable quantities depend on N_f in the crossover region; operationally, what we actually vary is α_*. In ref. [8] the range of α_* used for the calculation of meson masses was chosen to be $0.89 \le \alpha_* \le 1.0$, an interval where there is pronounced walking behavior. For the case $N_c = 3$ considered in ref. [8] and here, given the above-mentioned value, $\alpha_{cr} = \pi/4$, it follows that this lower limit, $\alpha_* = 0.89$, is about 12% greater than this critical coupling. The reason for this choice of lower limit on α_* was that the hadron masses become exponentially small relative to the scale Λ as $\alpha_* - \alpha_{cr} \to 0^+$, rendering numerical evaluations of the relevant integrals increasingly difficult in this limit. For our study of the shift away from walking behavior we consider an interval extending to larger couplings, from $\alpha_* = 1.0$ to $\alpha_* = 2.5$. Our upper limit is chosen in order for the ladder approximation used in our solutions of the Schwinger-Dyson and Bethe-Salpeter equations to have reasonable reliability. From eq. (2.1) it follows that $\alpha_* = 0.89$ corresponds to $N_f = 11.65$, about 2% less than $N_{f,cr}$ [15]. For a coupling as large as $\alpha_* = 2.5$, the semi-perturbative methods used to derive eqs. (2.1) are subject to sigificant corrections from higher-order perturbative, and from nonperturbative, contributions; recognizing this, the above upper limit of α_* corresponds formally to $N_f \simeq 9.8$, a roughly 20% reduction from $N_{f,cr} = 11.9$. Since the chiral transition which occurs as N_f increases through $N_{f,cr}$ is second-order (continuous), and since there is no spontaneous chiral symmetry breaking

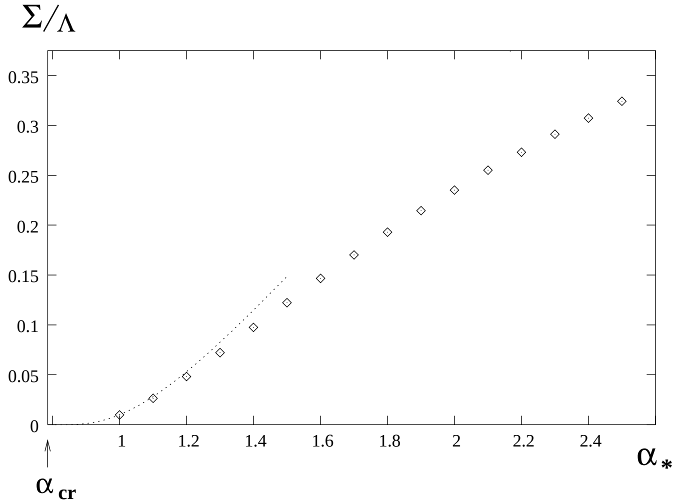

Figure 1: Numerical solutions for Σ, for several values of α_* (indicated by $\Diamond$). For comparison, we show eq. (3.1) with $c = 4.0$ from a fit to the results for $0.89 \leq \alpha_* \leq 1.0$.

in the non-Abelian Coulomb phase, it follows that as $N_f \nearrow N_{f,cr}$, (i) the masses of all hadron states vanish continuously; and (ii) hadron states that are related to each other by a parity reflection become degenerate.

3. Schwinger-Dyson equation

We first use the Schwinger-Dyson equation for the fermion propagator to calculate the dynamically generated mass Σ of this fermion. In figure 1 we show the solution for the dynamical fermion mass Σ as a function of α_*. A fit to the numerical solution in the walking region $0.89 \leq \alpha_* \leq 1.0$ [8] found agreement with the functional form

$$\Sigma = c\Lambda \, \exp\left[-\pi \Big(\frac{\alpha_*}{\alpha_{cr}} - 1 \Big)^{-1/2} \right], \tag{3.1}$$

with $c = 4.0$. An earlier analysis and numerical fit [5, 7] found agreement with $\Sigma \propto \Lambda \exp[-0.82\pi(\alpha_*/\alpha_{cr} - 1)^{-1/2}]$. Our calculations for larger α_* show the expected shift away from walking behavior. This shift is evident in figure 1 for α_* larger than about 1.2. In real QCD, precision fits to deep inelastic lepton scattering data, hadronic decays of the Z, etc. probe the theory in momentum regions where $N_f = 4$ or $N_f = 5$, and yield, for the effective N_f-dependent scale $\Lambda^{(5)}_{\rm QCD} \simeq 200\,{\rm MeV}$ and $\Lambda^{(4)}_{\rm QCD} \simeq 280\,{\rm MeV}$, with larger values for $\Lambda^{(N_f)}_{\rm QCD}$ with $N_f = 3, 2$. In actual QCD one thus has $\Sigma/\Lambda^{(N_f)} \simeq O(1)$ for these low values of N_f. These contrast with the limiting walking behavior, in which $\Sigma \ll \Lambda$, as indicated in eq. (3.1). Our calculation of Σ, shown in figure 1, shows that Σ/Λ increases substantially, by about a factor of 30, from a value of about 0.01 at $\alpha_* = 1.0$ to 0.32 at $\alpha_* = 2.5$, much closer to the value of O(1) for this ratio in QCD.

Another quantity of interest is the pseudoscalar decay constant f_P, the N_f-flavor generalization of the pion decay constant. For $N_f = 2$ QCD this is defined as $\langle 0|J^j_\mu|\pi^k(q)\rangle =$

$if_\pi q_\mu \delta^{jk}$ where $1 \le j,k \le 3$ are SU(2) isospin indices. Here, we use a generalization of this definition, with the symbol f_π replaced by f_P and the SU(N_f) isospin indices in the range $1 \le j,k \le N_f^2 - 1$. In QCD, one rough measure of the dynamical (constituent) quark mass is $\Sigma \simeq M_N/N_c \simeq 313\,\text{MeV}$, where M_N is the nucleon mass. An alternate definition sets $\Sigma \simeq M_\rho/2$; this would yield a somewhat larger value. Here we use the estimate $\Sigma \simeq 330\,\text{MeV}$. Using the value $f_\pi \simeq 92.4 \pm 0.3\,\text{MeV}$ [16], one thus has

$$\left(\frac{\Sigma}{f_\pi}\right)_{\rm QCD} \simeq 3.6 \; . \tag{3.2}$$

An approximate relation connecting Σ and f_P is [18] (with $y \equiv k_E^2$)

$$f_P^2 = \frac{N_c}{4\pi^2}\int_0^\infty y\,dy\,\frac{\Sigma^2(y) - \frac{y}{4}\frac{d}{dy}\left[\Sigma^2(y)\right]}{[y + \Sigma^2(y)]^2} \; . \tag{3.3}$$

The integration is rendered finite by the softness of the dynamical mass $\Sigma(k_E^2)$. Thus, the relation (3.3) suggests that for QCD, $f_\pi^2 \simeq N_c\Sigma^2/(4\pi^2)$. For $N_c = 3$, this is $\Sigma/f_\pi \simeq 2\pi/\sqrt{3} \simeq 3.6$, which agrees, to within the theoretical uncertainties, with experiment. In QCD, with $\Lambda^{(2)} \simeq 400\,\text{MeV}$, one has $f_\pi/\Lambda^{(2)}_{\rm QCD} \simeq 0.25$.

In figure 2 we show our results for f_P calculated from substituting our solution for $\Sigma(k^2)$ into eq. (3.3). In the walking limit, f_P has been shown to satisfy a relation similar to eq. (3.1), i.e., it is exponentially smaller than the scale Λ. We display, as the dotted curve, the fit from ref. [8] for the walking interval $0.89 \le \alpha_* \le 1.0$, given by eq. (3.1) with $c = 1.5$. Our results show the change from this walking type of behavior as α_* increases above 1.2; as α_* increases from 1.0 to 2.5, f_P/Λ increases substantially, from about 3×10^{-3} to about 0.08. This is similar to the factor by which we found that Σ/Λ increased as α_* increased through this interval.

The strong increase in Σ/Λ and f_P/Λ as α_* ascends from the value 0.89 near the walking limit to the value 2.5 deeper within the confinement phase is easily understood as reflecting the removal of the extreme exponential suppression evident in eq. (3.1) and its analogue for f_P for $\alpha_* - \alpha_{cr} \to 0^+$. One does not expect such a dramatic change in the ratio Σ/f_P over this interval, and this expectation is borne out by our calculations. In figure 3 we show the ratio of Σ/f_P, which increases gradually from about 2.6 to 3.9. The fact that we find a ratio comparable to the observed one in actual QCD, given by eq. (3.2), can be understood as a consequence of the property that much of the strong dependence on N_f divides out in this ratio.

4. Calculation of meson masses via the Bethe-Salpeter equation

4.1 General discussion

We denote the wavefunction for a hadron with a given flavor combination for the generalized π, ρ, etc. as follows. Define the flavor vector $f^a \equiv (f^{a1}, \ldots, f^{aN_f})$. Recall that in the confined phase the global symmetry $\text{SU}(N_f)_L \times \text{SU}(N_f)_R \times \text{U}(1)_V$ is broken spontaneously to $\text{SU}(N_f)_V \times \text{U}(1)_V$. We drop the explicit subscript V on $\text{SU}(N_f)_V$ henceforth.

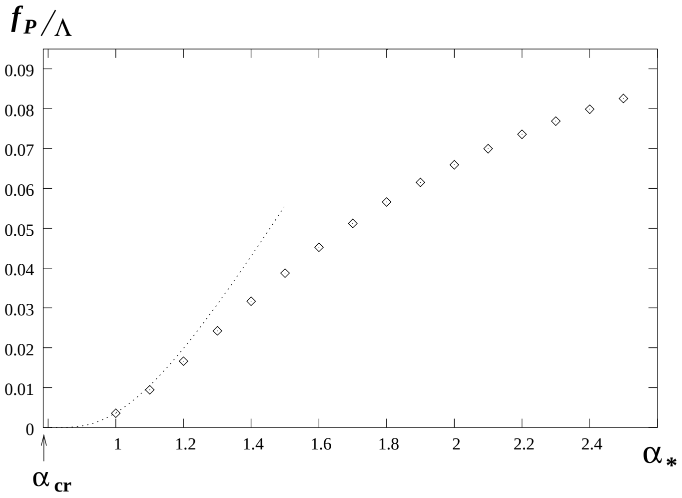

Figure 2: Values of f_P calculated from eq. (3.3) for several values of α_* (indicated by $\Diamond$). Dotted curve is eq. (3.1) with Σ replaced by f_P and $c = 1.5$ from a fit to the calculations for $0.89 \le \alpha_* \le 1.0$.

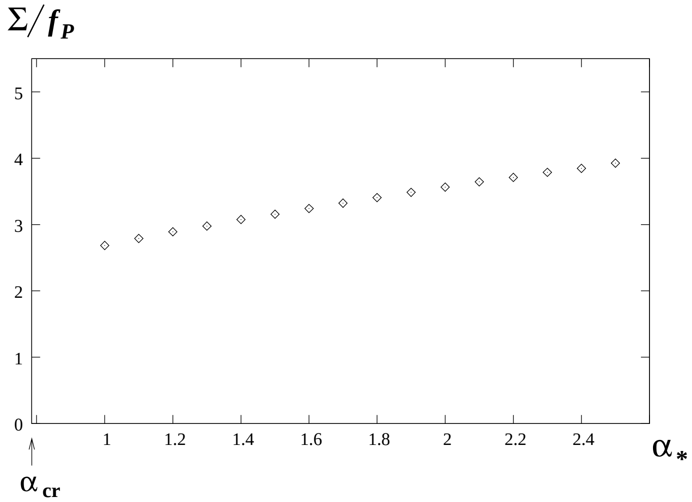

Figure 3: Plot of the ratio Σ/f_P for $1 \le \alpha_* \le 2.5$.

With regard to SU(N_f), a $f\bar{f}$ meson with a given $J^{\rm PC}$ (where J denotes the spin, and P and C are the parity and charge conjugate eigenvalues) is described via the Clebsch-Gordan decomposition $N_f \times \bar{N}_f = 1 + Adj$, where 1 and Adj denote the singlet and adjoint representations.

Let the generators of the group SU(N_f) have the standard normalization ${\rm Tr}(T_i T_j) = (1/2)\delta_{ij}$. Then the hadrons transforming according to the adjoint representation of SU(N_f) are comprised of (i) the set of $N_f(N_f - 1)$ states

$$h_{\Gamma;ij} = \frac{1}{\sqrt{N_c}} \sum_{a=1}^{N_c} \bar{f}_a \, \Gamma \, T_{ij} \, f^a \tag{4.1}$$

$n^{2S+1}L_J$	$J^{\rm PC}$	$R_{{\rm SU(2)}_V}$	name	M	M/f_π
1^3S_1	1^{--}	adj.	ρ	775.8 ± 0.5	8.40
1^3S_1	1^{--}	sing.	ω	782.6 ± 0.1	8.47
1^1P_1	1^{+-}	adj.	b_1	1229.5 ± 3.2	13.3
1^1P_1	1^{+-}	sing.	h_1	1170 ± 20	12.7 ± 0.2
1^3P_0	0^{++}	adj.	a_0	984.7 ± 1.2	10.7
1^3P_0	0^{++}	sing.	f_0	$\sim 600^{+600}_{-200}$	$6.5^{+6.5}_{-4.3}$
1^3P_1	1^{++}	adj.	a_1	1230 ± 40	13.3 ± 0.4
1^3P_1	1^{++}	sing.	f_1	1281.8 ± 0.6	13.9

Table 1: Data on relevant $q\bar{q}$ mesons whose masses are compared with Bethe-Salpeter calculations. $n^{2S+1}L_J$ is standard spectroscopic notation, where n denotes radial quantum number. The symbols adj. and sing. denote the adjoint and singlet representations of the ${\rm SU(2)}_V$ isospin flavor symmetry group. Masses are given in MeV. The last column lists the mass divided by a typical hadronic scale, f_π.

where T_{ij} is the $N_f \times N_f$ matrix with a 1 in the i'th column and j'th row, with $1 \le i, j \le N_f$, $i \ne j$, and Γ specifies the type of particle (pseudoscalar, vector, axial-vector, scalar), and (ii) the $N_f - 1$ states corresponds to the generators of the Cartan subalgebra of ${\rm SU}(N_f)$. Because of the ${\rm SU}(N_f)$ flavor symmetry, it does not matter which of these $N_f^2 - 1$ hadrons with a given Γ we use. We shall refer to these as the N_f-generalized ρ, ω, etc. In particular, the spectrum of mesons includes a set of $N_f^2 - 1$ Nambu-Goldstone bosons (NGB's) with $L = S = 0$ and $J^{\rm PC} = 0^{-+}$, transforming according to the adjoint representation of ${\rm SU}(N_f)$. The corresponding 0^{-+} singlet with respect to ${\rm SU}(N_f)$, i.e., the generalized η', is not a Nambu-Goldstone boson because the corresponding ${\rm U}(1)_A$ symmetry is anomalous. Our analysis of meson masses is for the lowest-lying $f\bar{f}$ states.

In QCD, there are several (light-quark) $\bar{q}q$ mesons that are of interest here. For the reader's convenience, we list these in table 1. A notation for the various states in the case of general N_f massless quarks is S_R, P_R, V_R, and A_R, standing for "scalar, pseudoscalar, vector, and axial-vector", where the subscript R denotes the representation - adjoint or singlet - under the ${\rm SU}(N_f)$ flavor symmetry group. The experimental and theoretical situation concerning the 0^{++} isoscalar meson f_0 has been the subject of much discussion over the years; indeed, this state may involve mixing with $qq\bar{q}\bar{q}$ mesons [19]. Because of the complications in the analysis of this state, and the expected complications in a realistic analysis of its N_f-generalization, the ${\rm SU}(N_f)$-singlet 0^{++} meson, we do not attempt to treat this in our current study.

In the Bethe-Salpeter equation that we use to calculate the masses of the mesons, the flavor-dependent structure is simply a prefactor. Hence, the solutions of this equation have the property that, for a given radial quantum number and spectroscopic form ${}^{2S+1}L_J$, the ${\rm SU}(N_f)$ flavor-singlet and flavor-adjoint mesons are degenerate:

$$M(n^{2S+1}L_J; \text{flav. adjoint}) = M(n^{2S+1}L_J; \text{flav. singlet}) \tag{4.2}$$

In view of this, we henceforth drop the subscript R and simply write V rather than $V_{flav.adj.}$ or $V_{flav.sing.}$, etc. Note that this is different from the prediction from ${\rm SU}(N_f)$ flavor symmetry (with degenerate quarks and electroweak interactions turned off) that the members of a given representation of $\mathrm{SU}(N_f)$ are degenerate. Experimentally, except for the pseudoscalar mesons, the light-quark isospin-adjoint and isospin-singlet $q\bar{q}$ mesons are nearly degenerate. The physical ω meson is very nearly a singlet under isospin SU(2), so a measure of this predicted degeneracy for the ground state 1^{--} mesons is $(M_\omega - M_\rho)/[(1/2)((M_\omega + M_\rho)] = 0.87 \times 10^{-2}$, quite small. Similarly, $(M_{f_1} - M_{a_1})/[(1/2)((M_{f_1} + M_{a_1})] = 0.04 \pm 0.03$ and $(M_{h_1} - M_{b_1})/[(1/2)((M_{h_1} + M_{b_1})] = -0.05 \pm 0.02$. So for these states the prediction from our Bethe-Salpeter technique for the special case of $N_f = 2$ massless quarks is in agreement with the data for light-quark mesons in QCD. In QCD, one has $M_{a_1}/M_\rho = 1.59 \pm 0.05$ and $M_{a_0}/M_\rho = 1.27$.

An interesting result of the calculations of meson masses in the walking limit in ref. [8] was that the ratios of these masses to f_P are rather constant. Specifically, it was found that in for $0.89 \leq \alpha_* \leq 1.0$, $M_V/f_P \simeq 11$, $M_A/f_P \simeq 11.5$, and $M_S/f_P \simeq 4.1$, so that $M_A/M_V = 1.04$ and $M_S/M_V = 0.36$, where the theoretical uncertainty is several per cent. These ratios may be compared with the values in regular QCD which, as far as the light-meson spectrum is concerned, are close to the values that they would have in the $N_f = 2$ chiral limit (with the understanding that the pion masses would actually vanish in this limit if electroweak interactions are turned off, as assumed here). For the purpose of this comparison, we do not try to use the inferred chiral-limit value of f_π [16], since to be consistent we would have to do the same for the mesons themselves. For the comparison between the extreme walking limit (WL) and QCD, we have

$$\frac{(M_V/f_P)_{\mathrm{WL}}}{(M_\rho/f_\pi)} \simeq 1.3 \tag{4.3}$$

$$\frac{(M_A/f_P)_{\mathrm{WL}}}{(M_{a_1}/f_\pi)} \simeq 0.86\,, \tag{4.4}$$

and

$$\frac{(M_S/f_P)_{\mathrm{WL}}}{(M_{a_0}/f_\pi)} \simeq 0.38\ . \tag{4.5}$$

A major output of the present work is the elucidation of how, as N_f decreases and α_* increases, the ratios of meson masses to f_P begin to shift toward their QCD values.

4.2 Numerical results

We next present the results of the numerical calculations for the masses of the mesons. We solve the homogeneous Bethe-Salpeter equation as an eigenvalue problem, namely, the Bethe-Salpeter amplitude as an eigenfunction and the mass of a bound state as an eigenvalue, denoted generically as M_B. First, as in ref. [8], we have checked and confirmed that the flavor-adjoint pseudoscalar meson mass is zero to within the numerical accuracy of our calculation. In figure 4, we show the values of meson masses divided by Λ calculated from the Schwinger-Dyson and Bethe-Salpeter equations in the range $0.9 \leq \alpha_* \leq 2.5$. In figure 5 we plot the values of M_B/f_P in the range of $0.9 \leq \alpha_* \leq 2.5$. In figure 6, we plot the meson mass ratios M_A/M_V and M_S/M_V in the range of $0.9 \leq \alpha_* \leq 2.5$.

Our calculations yield a number of interesting results. We summarize these for the changes in these meson masses as α_* increases from 0.9 to 2.5 as follows.

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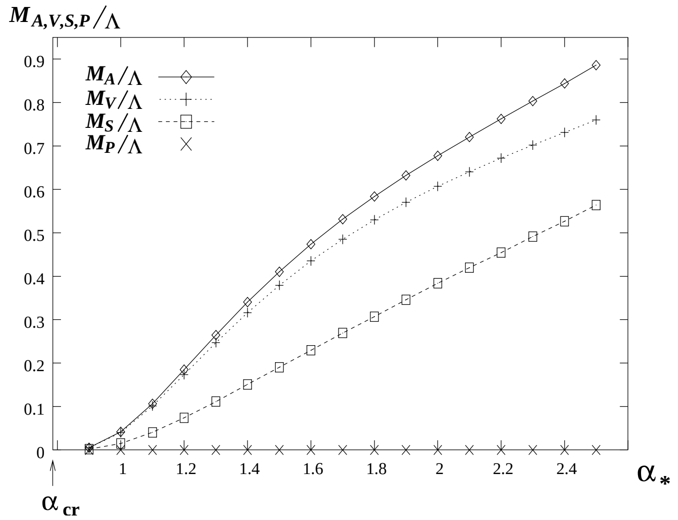

Figure 4: Values of meson masses divided by Λ calculated from the Schwinger-Dyson and Bethe-Salpeter equations.

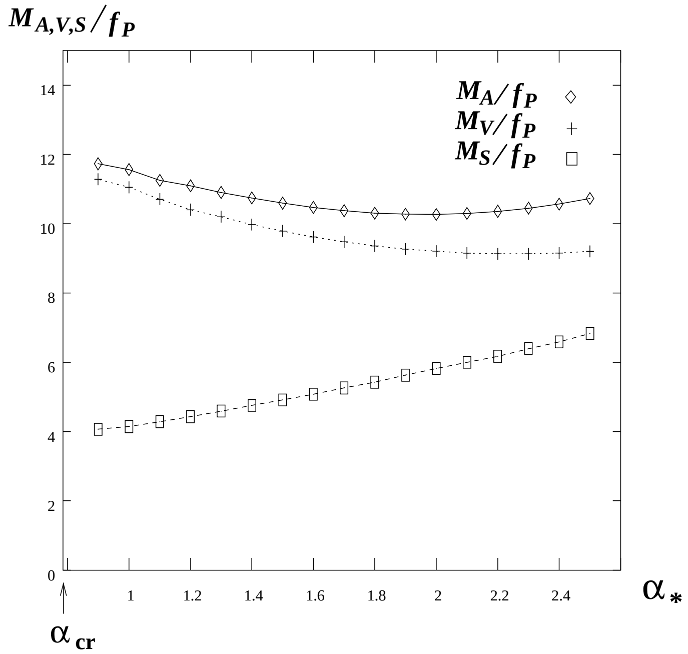

Figure 5: Values of meson masses divided by f_P calculated from the Schwinger-Dyson and Bethe-Salpeter equations.

1. The ratios of the meson masses divided by Λ increase dramatically, by factors of order 10^2, approaching values of order unity at $\alpha_* = 2.5$. This amounts to the removal of the exponential suppression of these masses which had described the walking limit at the boundary of the non-Abelian phase, as one moves away from this limit into the interior of the confined phase.

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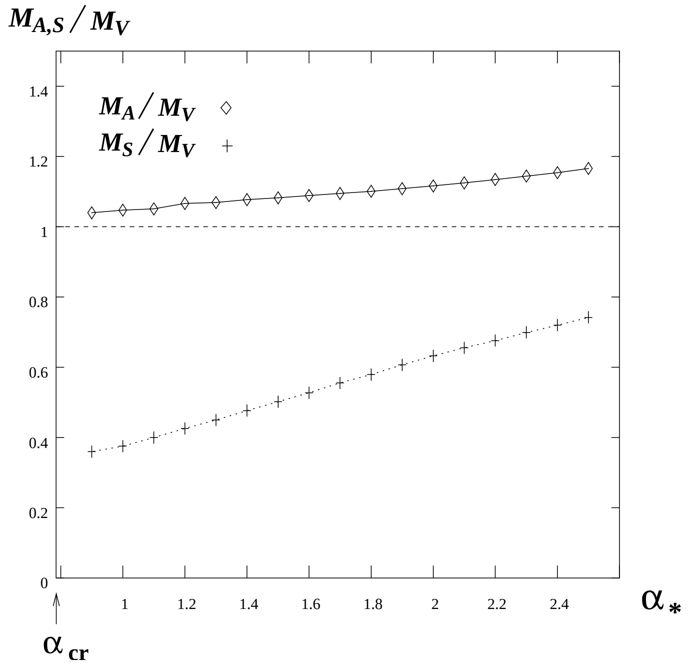

Figure 6: Ratios of meson masses calculated from the Schwinger-Dyson and Bethe-Salpeter equations.

2. M_S/f_P increases monotonically from about 4 to 7, thereby approaching to within about 35% of the value 10.7 in QCD for M_{a_0}/f_π.

3. M_V/f_P decreases from about 11 to 9, rather close to the value ~ 8.5 for M_ρ/f_π and M_ω/f_π in QCD. As is evident from figure 5, this ratio M_V/f_P is roughly constant in the upper end of the interval of α_* values that we study.

4. M_A/f_P behaves non-monotonically, first decreasing from roughly 11.5 to 10, but then increasing to about 11, within about 20% of the average of the values in QCD for the isospin-triplet and isospin-singlet axial-vector mesons, 13 for M_{a_1}/f_π and 14 for M_{f_1}/f_π.

5. Thus, the ratios M_A/M_V and M_S/M_V, which were found in ref. [8] to have the respective values 1.04 and 0.36 in the walking limit, both increase in the interval of α_* that we study, reaching about 1.17 and 0.74, respectively, at $\alpha_* = 2.5$. For comparison, these ratios are approximately 1.6 and 1.3 in QCD. Although the value of the ratio M_S/M_V at $\alpha_* = 2.5$ is farther from its QCD value than is the case with M_A/M_V, it is increasing somewhat more rapidly as a function of α_*, consistent with eventually approaching the QCD value.

5. Conclusions

In this paper we have considered a vectorial SU(N_c) gauge theory with N_f massless fermions transforming according to the fundamental representation and have studied the shift in behavior from walking that occurs in the region near the boundary between the confinement phase and the non-Abelian Coulomb phase to the QCD-like behavior with a non-walking coupling. Specifically, we have used the Schwinger-Dyson and Bethe-Salpeter equations to calculate the dynamically induced fermion mass Σ, the spontaneous chiral symmetry breaking parameter f_P, and the masses of the lowest-lying $q\bar{q}$ vector, axial-vector, and flavor-adjoint scalar mesons. We have investigated how these change as one decreases N_f below $N_{f,cr}$, or equivalently, increases α_* above α_{cr}, to move away from the above-mentioned boundary into the interior of the confinement phase. Our results show the crossover between walking and non-walking behavior in a gauge theory.

There are a number of interesting topics for future research using the methods of this paper. It would be useful to construct a kernel for the Bethe-Salpeter equation that could include more of the relevant physics, including instantons effects. Work is underway on this. It would also be worthwhile to calculate the masses of radially excited mesons and mesons with internal orbital angular momenta $L \geq 2$, as well as glueballs and the mixing between glueballs and $\bar{q}q$ mesons. We anticipate that the results of these calculations would exhibit the same general properties that we have found with the ground-state $\bar{q}q$ mesons, but it would be interesting to confirm this expectation explicitly. However, when one moves this far away from the walking regime, one loses a simplifying features of our calculation, namely the fact that we do not have to use an infrared cutoff on α. Given that lattice gauge theory methods provide an *ab initio* framework for calculating hadron masses, we hope that the lattice community will extend early efforts such as those of ref. [14] and carry out a definitive study of hadron masses in QCD with an arbitrary number of flavors. It would be of considerable interest to compare the results of the lattice calculations with those obtained from solutions of Schwinger-Dyson and Bethe-Salpeter equations.

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References

[1] An relevant early paper is T. Banks and A. Zaks, *On the phase structure of vector - like gauge theories with massless fermions*, *Nucl. Phys.* **B 196** (1982) 189.

[2] B. Holdom, *Techniodor*, *Phys. Lett.* **B 150** (1985) 301.

[3] K. Yamawaki, M. Bando and K.-I. Matumoto, *Scale invariant technicolor model and a technidilaton*, *Phys. Rev. Lett.* **56** (1986) 1335.

[4] T.W. Appelquist, D. Karabali and L.C.R. Wijewardhana, *Chiral hierarchies and the flavor changing neutral current problem in technicolor*, *Phys. Rev. Lett.* **57** (1986) 957;
T. Appelquist and L.C.R. Wijewardhana, *Chiral hierarchies and chiral perturbations in technicolor*, *Phys. Rev.* **D 35** (1987) 774; *Chiral hierarchies from slowly running couplings in technicolor theories*, *Phys. Rev.* **D 36** (1987) 568.

[5] T. Appelquist, J. Terning and L.C.R. Wijewardhana, *The zero temperature chiral phase transition in* SU(N) *gauge theories*, *Phys. Rev. Lett.* **77** (1996) 1214 [hep-ph/9602385].

[6] V.A. Miransky and K. Yamawaki, *Conformal phase transition in gauge theories*, *Phys. Rev.* **D 55** (1997) 5051 *Erratum ibid.* **56** (1997) 3768 [hep-th/9611142].

[7] T. Appelquist, A. Ratnaweera, J. Terning and L.C.R. Wijewardhana, *The phase structure of an* SU(N) *gauge theory with* N_f *flavors*, *Phys. Rev.* **D 58** (1998) 105017 [hep-ph/9806472].

[8] M. Harada, M. Kurachi and K. Yamawaki, *Meson masses in large-*N_f *QCD from Bethe-Salpeter equation*, *Phys. Rev.* **D 68** (2003) 076001 [hep-ph/0305018].

[9] M. Harada and Y. Yoshida, *Solving the homogeneous Bethe-Salpeter equation*, *Phys. Rev.* **D 53** (1996) 1482 [hep-ph/9505206].

[10] K.-I. Aoki, T. Kugo and M.G. Mitchard, *Meson properties from the ladder Bethe-Salpeter equation*, *Phys. Lett.* **B 266** (1991) 467.

[11] M. Harada, M. Kurachi and K. Yamawaki, $\pi^+ - \pi^0$ *mass difference from the Bethe-Salpeter equation*, *Phys. Rev.* **D 70** (2004) 033009 [hep-ph/0403120].

[12] M. Harada, M. Kurachi and K. Yamawaki, *Enhanced* $\pi^+ - \pi^0$ *mass difference and vanishing* S *in the large-*N_f *QCD*, *Prog. Theor. Phys.* **115** (2006) 765 [hep-ph/0509193].

[13] T. Appelquist, K.D. Lane and U. Mahanta, *On the ladder approximation for spontaneous chiral symmetry breaking*, *Phys. Rev. Lett.* **61** (1988) 1553;
T. Appelquist, U. Mahanta, D. Nash and L.C.R. Wijewardhana, *Gauge invariance of fermion masses in extended technicolor theories*, *Phys. Rev.* **D 43** (1991) 646;
T. Appelquist and S.B. Selipsky, *Instantons and the chiral phase transition*, *Phys. Lett.* **B 400** (1997) 364 [hep-ph/9702404].

[14] R. Mawhinney, *Review of unquenched results*, *Nucl. Phys.* **83** *(Proc. Suppl.)* (2000) 57;
Y. Iwasaki, K. Kanaya, S. Kaya, S. Sakai and T. Yoshie, *Phase structure of lattice QCD for general number of flavors*, *Phys. Rev.* **D 69** (2004) 014507 [hep-lat/0309159].

[15] Here and below, when we mention non-integral values of N_f, it is understood that physical values of N_F are, of course, non-negative integers.

[16] It has been estimated [17] that $f_\pi/(f_\pi)_{ch.lim.} \simeq 1.06$, where $(f_\pi)_{ch.lim.}$ denotes the value of f_π in the chiral limit $m_u = m_d = 0$. With $f_\pi = 92.4$, this gives $(f_\pi)_{ch.lim.} \simeq 87\,\mathrm{MeV}$. For our purposes, one may neglect this small difference between the actual f_π and the value $(f_\pi)_{ch.lim.}$

[17] J. Gasser and H. Leutwyler, *Chiral perturbation theory: expansions in the mass of the strange quark*, *Nucl. Phys.* **B 250** (1985) 465.

[18] H. Pagels and S. Stokar, *The pion decay constant, electromagnetic form-factor and quark electromagnetic selfenergy in QCD*, *Phys. Rev.* **D 20** (1979) 2947.

[19] D. Black, A.H. Fariborz, F. Sannino and J. Schechter, *Putative light scalar nonet*, *Phys. Rev.* **D 59** (1999) 074026 [hep-ph/9808415];
A.H. Fariborz, R. Jora and J. Schechter, *Generalized sigma model description of the light* $J = 0$ *mesons*, hep-ph/0601216.

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